Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | | | | | |

Candidate Number | | | | |

Pearson Edexcel International Advanced Level

Time 1 hour 30 minutes

Paper reference **WFM01/01**

Mathematics

International Advanced Subsidiary/Advanced Level
Further Pure Mathematics F1

You must have:
Mathematical Formulae and Statistical Tables (Yellow), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over ▶

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1.
$$\mathbf{A} = \begin{pmatrix} 3 & a \\ -2 & -2 \end{pmatrix}$$

where a is a non-zero constant and $a \neq 3$

(a) Determine $\mathbf{A}^{-1}$ giving your answer in terms of a.
(2)

Given that $\mathbf{A} + \mathbf{A}^{-1} = \mathbf{I}$ where $\mathbf{I}$ is the 2×2 identity matrix,

(b) determine the value of a.
(3)

a. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(x) = (a)(d) - (b)(c)$

$\det(A) = (3)(-2) - (a)(-2)$

$= (-6) - (-2a)$

$= -6 + 2a$

$\det(A) = 2a - 6$

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X^{-1} = \dfrac{1}{\det(x)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$-b = -(a) = -a$

$-c = -(-2) = 2$

$$\boxed{A^{-1} = \frac{1}{2a-6} \begin{bmatrix} -2 & -a \\ 2 & 3 \end{bmatrix}}$$

b. 2×2 identity matrix : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A + A^{-1} = I$

use ans from part (a)

$$\begin{bmatrix} 3 & a \\ -2 & -2 \end{bmatrix} + \frac{1}{2a-6} \begin{bmatrix} -2 & -a \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Question 1 continued

$$\begin{bmatrix} 3 & a \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} \frac{-2}{2a-6} & \frac{-a}{2a-6} \\ \frac{2}{2a-6} & \frac{3}{2a-6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 - \frac{2}{2a-6} & a - \frac{a}{2a-6} \\ -2 + \frac{2}{2a-6} & -2 + \frac{3}{2a-6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equate the elements and solve for a.

$$\begin{bmatrix} 3 - \frac{2}{2a-6} & a - \frac{a}{2a-6} \\ -2 + \frac{2}{2a-6} & -2 + \frac{3}{2a-6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$3 - \frac{2}{2a-6} = 1$

$2 = \frac{2}{2a-6}$

$1 = 2a - 6$

$2a = 7$

$a = \frac{7}{2}$

$\boxed{a = \frac{7}{2}}$

(Total 5 marks)

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Q1

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2. $$f(x) = 7\sqrt{x} - \frac{1}{2}x^3 - \frac{5}{3x} \qquad x > 0$$

(a) Show that the equation $f(x) = 0$ has a root, α, in the interval [2.8, 2.9]

(2)

(b) (i) Find $f'(x)$.

(ii) Hence, using $x_0 = 2.8$ as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to calculate a second approximation to α, giving your answer to 3 decimal places.

(4)

(c) Use linear interpolation once on the interval [2.8, 2.9] to find another approximation to α. Give your answer to 3 decimal places.

(3)

a. $f(2.8) = 7\sqrt{2.8} - \frac{1}{2}(2.8)^3 - \frac{5}{3(2.8)} = 0.1420022762$

$f(2.9) = 7\sqrt{2.9} - \frac{1}{2}(2.9)^3 - \frac{5}{3(2.9)} = -0.848642187 5$

There is a sign change from positive to negative from $f(2.8)$ to $f(2.9)$ and $f(x)$ is continuous therefore a root, α, lies in the interval [2.8, 2.9].

bi. $f(x) = 7\sqrt{x} - \frac{1}{2}x^3 - \frac{5}{3x}$

Rewrite $f(x)$ as indices:

$f(x) = 7x^{\frac{1}{2}} - \frac{1}{2}x^3 - \frac{5}{3}x^{-1}$

$f'(x) = \frac{7}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^2 + \frac{5}{3}x^{-2}$

$\boxed{f'(x) = \frac{7}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^2 + \frac{5}{3}x^{-2}}$

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Question 2 continued

bii. $X_0 = 2.8$

we are looking for X_1

Numerical Sol^{n}s of Eq^{n}s:

Newton-Rhapson iteration for solving $f(x)=0$

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

From Newton-Rhapson iteration: $X_1 = X_0 - \dfrac{f(X_0)}{f'(X_0)}$

(1) must find $f(X_0)$ by subbing in X_0 (1.4) into $f(x)$

(2) must find $f'(X_0)$ by differentiating $f(x)$ and subbing in X_0 (1.4) into $f'(x)$.

(1) $f(2.8) = 0.1420022762$ use part (a)

(2) $f'(x) = \frac{7}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^2 + \frac{5}{3}x^{-2}$

$f'(2.8) = \frac{7}{2}(2.8)^{-\frac{1}{2}} - \frac{3}{2}(2.8)^2 + \frac{5}{3}(2.8)^{-2} = -9.4557649$

$$X_1 = (2.8) - \frac{0.1420022762}{-9.4557649}$$

$X_1 = 2.815017535$

$\boxed{\alpha = 2.815 \quad (3 \text{ d.p.})}$

c. Linear Interpolation:

$\alpha = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$ for interval $[a,b]$

where α is the root

Question 2 continued

$a = 2.8$, $b = 2.9$

$$\alpha = \frac{2.8 f(2.9) - 2.9 f(2.8)}{f(2.9) - f(2.8)}$$

$f(2.8) = 0.1420022762$
$f(2.9) = -0.8486421875$

$\left.\right\}$ worked out from part (a)

$$\alpha = \frac{2.8(0.1420022762) - 2.9(-0.8486421875)}{(-0.8486421875) - (0.1420022762)}$$

$\alpha = 2.814334333$

$\boxed{\alpha = 2.814 \quad (3 \text{ d.p.})}$

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3. The quadratic equation

$$2x^2 - 5x + 7 = 0$$

has roots α and β

Without solving the equation,

(a) write down the value of $(\alpha + \beta)$ and the value of $\alpha\beta$

(1)

(b) determine, giving each answer as a simplified fraction, the value of

(i) $\alpha^2 + \beta^2$

(ii) $\alpha^3 + \beta^3$

(4)

(c) find a quadratic equation that has roots

$$\frac{1}{\alpha^2 + \beta} \text{ and } \frac{1}{\beta^2 + \alpha}$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers to be determined.

(4)

a. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

let α, β be roots of quadratic

$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$

sum of roots: $\alpha + \beta$

product of roots: $\alpha\beta$

$2x^2 - 5x + 7 = 0$ $\div 2$

$x^2 - \frac{5}{2}x + \frac{7}{2} = 0$

$\alpha + \beta = \frac{5}{2}$

$\alpha\beta = \frac{7}{2}$

$\boxed{\alpha + \beta = \frac{5}{2} \quad \alpha\beta = \frac{7}{2}}$

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Question 3 continued

bi. $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{5}{2}\right)^2 - 2\left(\frac{7}{2}\right)$

$= -\frac{3}{4}$

$\boxed{\alpha^2 + \beta^2 = -\frac{3}{4}}$

ii. $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= \left(\frac{5}{2}\right)^3 - 3\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)$

$= -\frac{85}{8}$

$\boxed{\alpha^3 + \beta^3 = -\frac{85}{8}}$

c. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - \left(\frac{1}{\alpha^2+\beta} + \frac{1}{\beta^2+\alpha}\right)x + \left(\left(\frac{1}{\alpha^2+\beta}\right)\times\left(\frac{1}{\beta^2+\alpha}\right)\right) = 0$

$x^2 - \left(\frac{\beta^2+\alpha+\alpha^2+\beta}{(\alpha^2+\beta)(\beta^2+\alpha)}\right)x + \left(\frac{1}{(\alpha^2+\beta)(\beta^2+\alpha)}\right) = 0$

$x^2 - \left(\frac{\alpha^2+\beta^2+\alpha+\beta}{\alpha^2\beta^2+\alpha^3+\beta^3+\alpha\beta}\right)x + \left(\frac{1}{\alpha^2\beta^2+\alpha^3+\beta^3+\alpha\beta}\right) = 0$

$x^2 - \left(\frac{\alpha^2+\beta^2+\alpha+\beta}{(\alpha\beta)^2+\alpha\beta+\alpha^3+\beta^3}\right)x + \left(\frac{1}{(\alpha\beta)^2+\alpha\beta+\alpha^3+\beta^3}\right) = 0$

use ans from part (bi) use ans from part (bii) use ans from part (bii)

$x^2 - \left(\frac{\left(-\frac{3}{4}\right)+\left(\frac{5}{2}\right)}{\left(\frac{7}{2}\right)^2+\left(\frac{7}{2}\right)+\left(-\frac{85}{8}\right)}\right)x + \left(\frac{1}{\left(\frac{7}{2}\right)^2+\left(\frac{7}{2}\right)+\left(-\frac{85}{8}\right)}\right) = 0$

$x^2 - \left(\frac{14}{41}\right)x + \left(\frac{8}{41}\right) = 0$

$x^2 - \frac{14}{41}x + \frac{8}{41} = 0$

$x^2 - \frac{14}{41}x + \frac{8}{41} = 0$ $\times 41$

$41x^2 - 14x + 8 = 0$ ← all integer coefficients

$\boxed{41x^2 - 14x + 8 = 0 \quad p = 41,\ q = -14,\ r = 8}$

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4. $$f(z) = 2z^3 - z^2 + az + b$$

where a and b are integers.

The complex number $-1 - 3i$ is a root of the equation $f(z) = 0$

(a) Write down another complex root of this equation. **(1)**

(b) Determine the value of a and the value of b. **(4)**

(c) Show all the roots of the equation $f(z) = 0$ on a single Argand diagram. **(2)**

a. For a cubic eq^n with form: $ax^3 + bx^2 + cx + d$, there are 3 roots.
There are 2 possibilities: 3 real roots
1 real root and 2 complex roots (conjugate pair)

The other complex root must be the complex conjugate of $(-1-3i)$
(flip sign of imaginary component

$\boxed{-1+3i}$

b. $(z-(-1-3i))(z-(-1+3i))(z-\gamma)$

$(z+1+3i)(z+1-3i)(z-\gamma)$

$(z^2 + z - \cancel{3iz} + z + 1 - \cancel{3i} + \cancel{3iz} + \cancel{3i} - 9i^2)(z-\gamma)$

$(z^2 + 2z + 1 - 9(-1))(z-\gamma)$

$(z^2 + 2z + 10)(z-\gamma)$

$z^3 + 2z^2 + 10z - \gamma z^2 - 2\gamma z - 10\gamma$

$(1)z^3 + (2-\gamma)z^2 + (10-2\gamma)z + (-10\gamma)$

$(2)z^3 + (4-2\gamma)z^2 + (20-4\gamma)z + (-20\gamma)$

equate coefficients to $f(z)$: (1) $2 = 2$
(2) $4 - 2\gamma = -1$
(3) $20 - 4\gamma = a$
(4) $-20\gamma = b$

use (2) to solve γ: $4 - 2\gamma = -1$
$5 = 2\gamma$
$\gamma = \frac{5}{2}$

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Question 4 continued

Sub $r = \frac{5}{2}$ into (3) and (4) to find a and b.

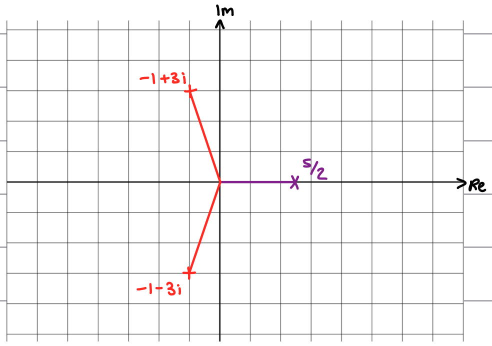
(3) $20 - 4r = a$ (4) $-20r = b$

$20 - 4\left(\frac{5}{2}\right) = a$ $-20\left(\frac{5}{2}\right) = b$

$a = 10$ $b = -50$

$\boxed{a = 10, b = -50}$

c.



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5. (a) Use the standard results for $\sum_{r=1}^{n} r^3$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that for all positive integers n,

$$\sum_{r=1}^{n} r(r-1)(r-3) = \frac{1}{12}n(n+1)(n-1)(3n-10)$$

(5)

(b) Hence show that

$$\sum_{r=n+1}^{2n+1} r(r-1)(r-3) = \frac{1}{12}n(n+1)(an^2+bn+c)$$

where a, b and c are integers to be determined.

(3)

a. $\sum_{r=1}^{n} r(r-1)(r-3) = \sum_{r=1}^{n} r(r^2-4r+3) = \sum_{r=1}^{n} r^3 - 4r^2 + 3r$

can split summation into 3

$\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} 4r^2 + \sum_{r=1}^{n} 3r$

factor 4 out summation

factor 3 out summation

$= \sum_{r=1}^{n} r^3 - 4\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r$

Summations

$\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

must learn!

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

in the formulae booklet

$= \frac{1}{4}n^2(n+1)^2 - 4\left[\frac{1}{6}n(n+1)(2n+1)\right] + 3\left[\frac{1}{2}n(n+1)\right]$

$= \frac{1}{4}n^2(n+1)^2 - \frac{4}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1)$

$= \frac{1}{4}n^2(n+1)^2 - \frac{2}{3}n(n+1)(2n+1) + \frac{3}{2}n(n+1)$

$= \frac{1}{12}n(n+1)\left[3n(n+1) - 8(2n+1) + 18\right]$

$= \frac{1}{12}n(n+1)\left[3n^2 + 3n - 16n - 8 + 18\right]$

$= \frac{1}{12}n(n+1)\left[3n^2 - 13n + 10\right]$

$\boxed{\frac{1}{12}n(n+1)(n-1)(3n-10) \;// \quad \text{shown}}$

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Question 5 continued

$$\sum_{r=n+1}^{2n+1} r(r-1)(r-3) = \sum_{r=1}^{2n+1} r(r-1)(r-3) - \sum_{r=1}^{n} r(r-1)(r-3)$$

$$= \frac{1}{12}(2n+1)(2n+1+1)(2n+1-1)(3[2n+1]-10) - \frac{1}{12}n(n+1)(n-1)(3n-10)$$

$$= \frac{1}{12}(2n+1)\underbrace{(2n+2)}_{\text{factor out } 2}(2n)(6n+3-10) - \frac{1}{12}n(n+1)(n-1)(3n-10)$$

$$= \frac{1}{12}(2n+1)\,2\,(n+1)(2n)(6n-7) - \frac{1}{12}n(n+1)(n-1)(3n-10)$$

$$= \frac{4}{12}n(n+1)(2n+1)(6n-7) - \frac{1}{12}n(n+1)(n-1)(3n-10)$$

factor out $\frac{1}{12}n(n+1)$

$$= \frac{1}{12}n(n+1)\left[4(2n+1)(6n-7) - (n-1)(3n-10)\right]$$

$$= \frac{1}{12}n(n+1)\left[4(12n^2-8n-7) - (3n^2-13n+10)\right]$$

$$= \frac{1}{12}n(n+1)\left[48n^2-32n-28-3n^2+13n-10\right]$$

$$= \frac{1}{12}n(n+1)\left[45n^2-19n-38\right]$$

$\boxed{\frac{1}{12}n(n+1)(45n^2-19n-38) \quad a=45,\ b=-19,\ c=-38}$

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6. The curve H has equation

$$xy = a^2 \qquad x > 0$$

where a is a positive constant.

The line with equation $y = kx$, where k is a positive constant, intersects H at the point P

(a) Use calculus to determine, in terms of a and k, an equation for the tangent to H at P **(4)**

The tangent to H at P meets the x-axis at the point A and meets the y-axis at the point B

(b) Determine the coordinates of A and the coordinates of B, giving your answers in terms of a and k **(2)**

(c) Hence show that the area of triangle AOB, where O is the origin, is independent of k **(2)**

a. find point P by solving where line eqn meets curve H simultaneously.

$y = kx$

$xy = a^2$

sub in (1) into (2)

$x(kx) = a^2$

$kx^2 = a^2$

$x^2 = \frac{a^2}{k}$

$x = \pm\sqrt{\frac{a^2}{k}} = \pm\frac{\sqrt{a^2}}{\sqrt{k}} = \pm\frac{a}{\sqrt{k}}$

However, since $x > 0$ must reject $x = \frac{-a}{\sqrt{k}}$

$\therefore x = \frac{a}{\sqrt{k}}$

sub into either (1) or (2) to find y

easier to sub into (1)

$y = k\left(\frac{a}{\sqrt{k}}\right)$

$y = \frac{ak}{k^{1/2}}$

$y = ak^{\frac{1}{2}} = a\sqrt{k}$

Point P: $\left(\frac{a}{\sqrt{k}}, a\sqrt{k}\right)$

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Question 6 continued

Differentiate curve H w.r.t. x:

$xy = a^2$

$y = \frac{a^2}{x} = a^2x^{-1}$

$\frac{dy}{dx} = -a^2x^{-2} = \frac{-a^2}{x^2}$

Sub in coord. $P\left(\frac{a}{\sqrt{k}}, a\sqrt{k}\right)$

$\left.\frac{dy}{dx}\right|_{\left(\frac{a}{\sqrt{k}}, a\sqrt{k}\right)} \quad \frac{-a^2}{\left(\frac{a}{\sqrt{k}}\right)^2} = \frac{-a^2}{\frac{a^2}{k}} = \frac{-a^2k}{a^2} = -k$

$M_{tangent} = -k$

set up line eq^n $y - y_1 = m(x - x_1)$ with $m = -k$

$(x_1, y_1) = \left(\frac{a}{\sqrt{k}}, a\sqrt{k}\right)$

$y - a\sqrt{k} = -k\left(x - \frac{a}{\sqrt{k}}\right)$

$y - a\sqrt{k} = -kx + a\sqrt{k}$

$y = -kx + 2a\sqrt{k}$

$\boxed{y = -kx + 2a\sqrt{k}}$

b. (1) @ x-axis, $y = 0$

(2) @ y-axis, $x = 0$

(1) $0 = -kx + 2a\sqrt{k}$

$kx = 2a\sqrt{k}$

$x = \frac{2a\sqrt{k}}{k} = \frac{2ak^{1/2}}{k} = \frac{2a}{k^{1/2}} = \frac{2a}{\sqrt{k}}$

$\boxed{A\left(\frac{2a}{\sqrt{k}}, 0\right)}$

(2) $y = -k(0) + 2a\sqrt{k}$

$y = 2a\sqrt{k}$

$\boxed{B\left(0, 2a\sqrt{k}\right)}$

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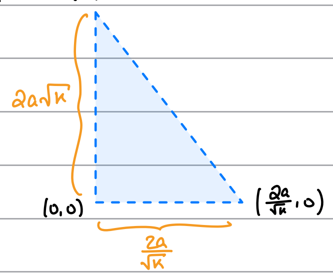
Question 6 continued

C. $(0, 2a\sqrt{k})$



$\Delta AOB = \frac{1}{\cancel{2}} \times \cancel{2}a\cancel{\sqrt{k}} \times \frac{2a}{\cancel{\sqrt{k}}}$

$= 2a^2$

$\boxed{\Delta AOB = 2a^2 \text{ units}^2}$

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7. In part (i), the elements of each matrix should be expressed in exact numerical form.

(i) (a) Write down the 2×2 matrix that represents a rotation of 210° anticlockwise about the origin.

(1)

(b) Write down the 2×2 matrix that represents a stretch parallel to the y-axis with scale factor 5

(1)

The transformation T is a rotation of 210° anticlockwise about the origin followed by a stretch parallel to the y-axis with scale factor 5

(c) Determine the 2×2 matrix that represents T

(2)

(ii)

$$\mathbf{M} = \begin{pmatrix} k & k+3 \\ -5 & 1-k \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Find det $\mathbf{M}$, giving your answer in simplest form in terms of k.

(2)

A closed shape R is transformed to a closed shape R' by the transformation represented by the matrix $\mathbf{M}$.

Given that the area of R is 2 square units and that the area of R' is $16k$ square units,

(b) determine the possible values of k.

(3)

a. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ← Anticlockwise rotation through θ about O (Given in Formulae booklet)

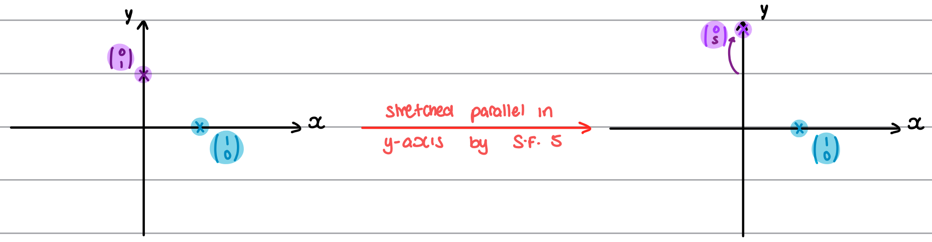
sub in $\theta = 210°$ → $\begin{bmatrix} \cos(210°) & -\sin(210°) \\ \sin(210°) & \cos(210°) \end{bmatrix}$ → $\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

$\boxed{\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}}$

Question 7 continued

b. Start by drawing a coordinate axis and mark coordinates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

then transform both of these coordinates by transformation (stretched parallel in y-axis by S.F. 5)



identity matrix I

rewrite matrix but now replace coordinates with their transformations in this specific order

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \xrightarrow{\text{new matrix represents transformation}} \begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}$$

$$\boxed{Q : \begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}}$$

c. $T = \begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}\begin{bmatrix}-\frac{\sqrt{3}}{2} & \frac{1}{2}\\-\frac{1}{2} & -\frac{\sqrt{3}}{2}\end{bmatrix}$

$$T = \begin{bmatrix}(1)(-\frac{\sqrt{3}}{2}) + (0)(-\frac{1}{2}) & (1)(\frac{1}{2}) + (0)(-\frac{\sqrt{3}}{2})\\(0)(-\frac{\sqrt{3}}{2}) + (5)(-\frac{1}{2}) & (0)(\frac{1}{2}) + (5)(-\frac{\sqrt{3}}{2})\end{bmatrix}$$

$$\boxed{T = \begin{bmatrix}-\frac{\sqrt{3}}{2} & \frac{1}{2}\\-\frac{5}{2} & \frac{-5\sqrt{3}}{2}\end{bmatrix}}$$

iia.

$$\boxed{\begin{gathered}X = \begin{bmatrix}a & b\\c & d\end{bmatrix}\\ \det(X) = (a)(d) - (b)(c)\end{gathered}}$$

$$\begin{aligned}\det(M) &= (k)(1-k) - (k+3)(-5)\\ &= k - k^2 + 5(k+3)\\ &= -k^2 + k + 5k + 15\\ &= -k^2 + 6k + 15\end{aligned}$$

$$\boxed{\det(M) = -k^2 + 6k + 15}$$

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Question 7 continued

b. $|\det(M)|$ = area scale factor

Area S.F. = $|\det(M)| = |-k^2 + 6k + 15|$

Area of $R \times |-k^2+6k+15|$ = Area of R'

$2 \times |-k^2 + 6k + 15| = 16k$

$|-k^2 + 6k + 15| = 8k$

$-k^2 + 6k + 15 = 8k$ or $-(-k^2+6k+15) = 8k$

$k^2 + 2k - 15 = 0$ $\qquad$ $k^2 - 6k - 15 = 8k$

$(k-3)(k+5) = 0$ $\qquad$ $k^2 - 14k - 15 = 0$

$k = 3$ or $k = -5$ $\qquad$ $(k-15)(k+1) = 0$

$k = 15$ or $k = -1$

$\boxed{k = -5, -1, 3, 15}$

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8. The parabola C has equation $y^2 = 20x$

The point P on C has coordinates $(5p^2, 10p)$ where p is a non-zero constant.

(a) Use calculus to show that the tangent to C at P has equation

$$py - x = 5p^2$$

(3)

The tangent to C at P meets the y-axis at the point A.

(b) Write down the coordinates of A.

(1)

The point S is the focus of C.

(c) Write down the coordinates of S.

(1)

The straight line l_1 passes through A and S.

The straight line l_2 passes through O and P, where O is the origin.

Given that l_1 and l_2 intersect at the point B,

(d) show that the coordinates of B satisfy the equation

$$2x^2 + y^2 = 10x$$

(5)

a. Differentiate parabola w.r.t. x:

$y^2 = 20x$

$2y\left(\frac{dy}{dx}\right) = 20$

$\frac{dy}{dx} = \frac{20}{2y} = \frac{10}{y}$

Sub in coord. C $(5p^2, 10p)$

$\left.\frac{dy}{dx}\right|_{(5p^2,10p)} \quad \frac{10}{(10p)} = \frac{1}{p}$

$M_{tangent} = \frac{1}{p}$

set up line eq^n $y - y_1 = m(x - x_1)$ with $m = \frac{1}{p}$

$(x_1, y_1) = (5p^2, 10p)$

Question 8 continued

$y - 10p = \frac{1}{p}(x - 5p^2)$) multiply both sides by 4.

$py - 10p^2 = x - 5p^2$

$\boxed{py - x = 5p^2}$ // shown

b. @ y - axis , $x = 0$

$py - (0) = 5p^2$

$py = 5p^2$

$y = \frac{5p^2}{p} = 5p$

$\boxed{\therefore A(0, 5p)}$

c. $y^2 = 20x$

$y^2 = 4(5)x$

$\therefore a = 5$

$\boxed{\text{focus } (5, 0)}$

d. (1) find line eq^n ℓ_1.

$m = \frac{5p - 0}{0 - 5} = \frac{5p}{-5} = -p$

$c = 5p$ (from coord. A $(0, 5p)$)

ℓ_1: $y = -px + 5p$

(2) find line eq^n ℓ_2.

$m = \frac{10p - 0}{5p^2 - 0} = \frac{10p}{5p^2} = \frac{2}{p}$

$c = 0$ (from origin $(0,0)$)

ℓ_2: $y = \frac{2}{p}x$

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Question 8 continued

(3) since ℓ_1 and ℓ_2 intersect, solve simultaneously

ℓ_1: $y = -px + 5p$

ℓ_2: $y = \frac{2}{p}x$

$-px + 5p = \frac{2}{p}x$

$5p = \frac{2}{p}x + px$

$5p^2 = 2x + p^2x$

$5p^2 = x(2+p^2)$

$x = \dfrac{5p^2}{p^2+2}$

sub back into either ℓ_1 or ℓ_2 to find y-coord:

$y = \dfrac{2}{\cancel{p}}\left(\dfrac{5p^{\cancel{2}}}{p^2+2}\right) = \dfrac{10p}{p^2+2}$

$\therefore B\left(\dfrac{5p^2}{p^2+2}, \dfrac{10p}{p^2+2}\right)$

(4) let $X = \dfrac{5p^2}{p^2+2}$ and $Y = \dfrac{10p}{p^2+2}$

calculate $2x^2 + y^2$

$2\left(\dfrac{5p^2}{p^2+2}\right)^2 + \left(\dfrac{10p}{p^2+2}\right)^2$

$= \dfrac{2(5p^2)^2}{(p^2+2)^2} + \dfrac{(10p)^2}{(p^2+2)^2} = \dfrac{2(25p^4) + 100p^2}{(p^2+2)^2} = \dfrac{50p^4 + 100p^2}{(p^2+2)^2}$

$= \dfrac{50p^2\cancel{(p^2+2)}}{(p^2+2)^{\cancel{2}}} = \dfrac{50p^2}{p^2+2}$

$\dfrac{50p^2}{p^2+2} = 10\left(\dfrac{5p^2}{p^2+2}\right) = 10x$

$\boxed{\therefore 2x^2 + y^2 = 10x_{//} \text{ shown}}$

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9. (i) A sequence of numbers is defined by

$$u_1 = 0 \qquad u_2 = -6$$

$$u_{n+2} = 5u_{n+1} - 6u_n \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 3 \times 2^n - 2 \times 3^n$$

(5)

(ii) Prove by induction that, for all positive integers n,

$$f(n) = 3^{3n-2} + 2^{4n-1}$$

is divisible by 11

(5)

i. (1) Base case: check true for $n = 1$ & $n = 2$

$u_1 = 3 \times 2^1 - 2 \times 3^1 = 0$

$u_2 = 3 \times 2^2 - 2 \times 3^2 = -6$

$u_1 = 0$ and $u_2 = -6$

$\therefore$ true for $n = 1$ and $n = 2$

(2) Assume true for $n = k$ & $n = k+1$:

$u_k = 3 \times 2^k - 2 \times 3^k$

$u_{k+1} = 3 \times 2^{k+1} - 2 \times 3^{k+1}$

(3) Induction step, prove true for $n = k+2$:

$u_{k+2} = 5u_{k+1} - 6u_k$

Sub in u_{k+1} and u_k from assumption step.

$u_{k+2} = 5\left[3 \times 2^{k+1} - 2 \times 3^{k+1}\right] - 6\left[3 \times 2^k - 2 \times 3^k\right]$

$u_{k+2} = \left[15 \times 2^{k+1} - 10 \times 3^{k+1}\right] - \left[18 \times 2^k - 12 \times 3^k\right]$

$= \left[15 \times (2^1)(2^k) - 10 \times (3^1)(3^k)\right] - \left[18 \times 2^k - 12 \times 3^k\right]$

$= \left[30(2^k) - 30(3^k)\right] - \left[18(2^k) - 12(3^k)\right]$

$= 30(2^k) - 30(3^k) - 18(2^k) + 12(3^k)$

$= 12(2^k) - 18(3^k)$

$= 3 \times 4 \times 2^k - 2 \times 9 \times 3^k$

Thinking ahead, sub in $n = k+2$ into u_n - this is what we are trying to prove using (2):

$u_{k+2} = 3 \times 2^{(k+2)} - 2 \times 3^{(k+2)}$

Make a note to the side so we know what we are working towards.

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Question 9 continued

$u_{k+2} = 3 \times 2^2 \times 2^k - 2 \times 3^2 \times 3^k$

$= 3 \times (2^2)(2^k) - 2 \times (3^2)(3^k)$

$= 3 \times 2^{k+2} - 2 \times 3^{k+2}$ $\therefore$ true for $n = k+2$

(4) Conclusion

If the result is true for $n=k$ and $n=k+1$ then it is true for $n=k+2$. As the result has been shown to be true for $n=1$ and $n=2$ then the result is true for all n.

ii. (1) Base Case: check true for $n=1$

$f(1) = 3^{3(1)-2} + 2^{4(1)-1}$

$= 11$

$= 1(11)$ ← divisible by 11

$\therefore$ true for $n=1$

(2) Assume true for $n=k$:

$f(k) = 3^{3k-2} + 2^{4k-1}$

(3) Inductive Step, Prove true for $n=k+1$:

$f(k+1) = 3^{3(k+1)-2} + 2^{4(k+1)-1}$

$= 3^{3k+3-2} + 2^{4k+4-1}$

$= 3^{3k+1} + 2^{4k+3}$

$f(k+1) - f(k) = (3^{3k+1} + 2^{4k+3}) - (3^{3k-2} + 2^{4k-1})$

$= [(3^{3k-2})(3^3) + (2^{4k-1})(2^4)] - (3^{3k-2} + 2^{4k-1})$

$= 27(3^{3k-2}) + 16(2^{4k-1}) - (3^{3k-2}) - (2^{4k-1})$

$= 26(3^{3k-2}) + 15(2^{4k-1})$

$= 11(3^{3k-2}) + 15(3^{3k-2}) + 15(2^{4k-1})$

$= 11(3^{3k-2}) + 15[3^{3k-2} + 2^{4k-1}]$

$= 11(3^{3k-2}) + 15f(k)$

$\therefore f(k+1) = 11(3^{3k-2}) + 16f(k)$ $\therefore$ true for $n=k+1$

Leave blank

Question 9 continued

(4) Conclusion

If this result is true for $n = k$, then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.

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